# Finite Size of the Electron and Runaway Solutions

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The problem of runaway solutions is studied within the framework of a non-local equation of motion for the classically radiating electron. It is found that the force-free electron oscillates down to a constant velocity under emission of radiation, if certain restrictions on the initial conditions are imposed. Causality violation is not present in this model, but penetrates into the theory as consequence of a false perturbation expansion leading to the notorious Lorentz-Dirac equation of motion.

## I. Introduction and Survey of Results

There are numerous attempts in literature to overcome the problems of radiation reaction by resorting to a finite-size model of the classically radiating electron. As a starting point for further studies of the literature, two recent papers [1,2] are recommended, which provide also the basis of the present work. The main idea elaborated in these two papers consists in the assumption that the classical electron is built up of two constituents: a purely mechanical one, characterized by the mechanical mass $m_{\mathrm{mech}}$, and a purely electromagnetic one, characterized by $m_{\mathrm{el}}$.

These two components are coupled together and capable of oscillations relative to each other, where the emission of radiation by the electromagnetic subsystem guarantees that these oscillations are damped down with increasing time. The basic equation of this two-component model of the radiating electron is

$$m_{\mathrm{mech}}\, c^2\, \dot{u}^\lambda + m_{\mathrm{el}}\, c^2\, \{\hat{\hat{u}}^\lambda - (u\,\hat{\hat{u}})\,u^\lambda\} = K^\lambda, \quad (\mathrm{I},1\mathrm{a})$$

or in its one-dimensional form

$$(\{u^\lambda\} \Rightarrow \{\mathrm{Cosh}\, w_{(s)};\ 0,\, 0,\, \mathrm{Sinh}\, w_{(s)}\})$$

$$m_{\mathrm{mech}}\, c^2\, \dot{w}_{(s)} + m_{\mathrm{el}}\, c^2\, \hat{w}\, \mathrm{Cosh}\, [w - \hat{w}] = K_{(s)}, \quad (\mathrm{I},1\mathrm{b})$$

where $K^\lambda$ denotes the external force acting on the electron as a whole. The contribution of the mechanical constituent appears as the derivative of the usual four-momentum encountered in relativistic point dynamics

$$m_{\mathrm{mech}}\, c^2\, \dot{u}^\lambda{}_{(s)} \equiv c\, \mathrm{d}P^\lambda_{\mathrm{mech}}/\mathrm{d}s \qquad (\mathrm{I},2\mathrm{a})$$

$$P^\mu_{\mathrm{mech}(s)} = m_{\mathrm{mech}}\, c\, u^\mu{}_{(s)}. \qquad (\mathrm{I},2\mathrm{b})$$

But the electromagnetic part enters the equation of motion in a more complicated way: The electromagnetic inertia

$$P^\mu_{\mathrm{el}} = m_{\mathrm{el}}\, c\, u^\mu \qquad (\mathrm{I},3)$$

runs behind the mechanical inertia by the constant amount $\Delta s$ ($\hat{\hat{u}}^\lambda := \dot{u}^\lambda{}_{(s-\Delta s)}$; $\hat{w} = w_{(s-\Delta s)}$), because one assumes this part of the inertia to be distributed in the immediate surroundings of the point-like mechanical mass so that it takes a characteristic proper time interval ($\Delta s$) for the mechanical mass to drag along the electromagnetic part, if an external force is accelerating the particle. Clearly, the information to be under the influence of an external force and to be consequently accelerated takes a finite time to travel through the extended structure, according to basic assumptions of Special Relativity.

The last term in (I, 1a) to be explained is, of course, identified with the radiated four-momentum [3]

$$c\, \mathrm{d}P^\mu_{\mathrm{rad}}/\mathrm{d}s = -\, m_{\mathrm{el}}\, c^2\, (u\,\hat{\hat{u}})\,u^\mu. \qquad (\mathrm{I},4)$$

Now we can formulate our problem to be treated in this paper: Since we know [2] that Eq. (I, 1) leads to the notorious Lorentz-Dirac equation

$$m\, c^2\, \hat{u}^\lambda = K^\lambda + \tfrac{2}{3}\, Z^2\, \{\ddot{u}^\lambda + (\dot{u}\,\dot{u})\,u^\lambda\} \qquad (\mathrm{I},5)$$

with its unphysical runaway solutions, if one expands the shifted quantities with respect to the nonlocality parameter $\Delta s$, one can ask, whether these runaway solutions arise as consequence of a faulty perturbation expansion or whether they are already contained in the "exact" equation (I, 1)? Moreover, we are interested in causality violation; is it

already contained in the original equation (I, 1) or is it penetrating by the same inconsistent perturbation expansion responsible for runaway solutions?

In the following, we shall be able to show that the force free particle obeying the equation of motion (I, 1) exhibits a certain stability against external disturbances: If the velocity is prescribed arbitrarily in a proper time interval of length $\Delta s$, then the particle performs damped oscillations around some finite equilibrium velocity, which is finally assumed by the particle. Only if the velocity differences during the initial $\Delta s$-interval are exceeding certain limits, depending from the mass ratio $m_{\mathrm{el}}/m_{\mathrm{mech}}$, then the particle works up to the velocity of light in a finite proper time.

As for causality violation, we shall find that this (at least in the domain of classical physics) unphysical phenomenon is not present in our improved theory based upon (I, 1).

We restrict ourselves to the one-dimensional case (I, 1b).

## II. Qualitative Discussion of the "Free Solutions"

In order to get some preliminary impression of the qualitative character of the solutions in the force free case where the equation of motion reads $(a := m_{\mathrm{el}}/m_{\mathrm{mech}})$

$$\dot{w}_{(s)} + a \, \dot{w}_{(s-\Delta s)} \operatorname{Cosh} \{w_{(s)} - w_{(s-\Delta s)}\} = 0, \quad \text{(II, 1)}$$

one can differentiate Eq. (II, 1) to obtain (assume for the moment that $w_{(s)}$ be of class $C^2[0, \infty]$)

$$\ddot{w}_{(s)} + a \, \ddot{w}_{(s-\Delta s)} \operatorname{Cosh} \Delta w$$
$$+ a \, \dot{w}_{(s-\Delta s)} \operatorname{Sinh} \Delta w \cdot \Delta \dot{w} = 0 \quad \text{(II, 2)}$$
$$(\Delta w := w_{(s)} - w_{(s-\Delta s)}).$$

From these two equations the following statements are easily established:

(1) Since the sign of $\dot{w}_{(s)}$ is opposite to that of $\dot{w}_{(s-\Delta s)}$ there must be at least one zero of $\dot{w}_{(s)}$. But this zero is reproduced equidistantly in $s$ on account of the proportionality between $\dot{w}_{(s)}$ and $\dot{w}_{(s-\Delta s)}$, so that the "free solutions" must have oscillatory character.

(2) If $\dot{w}_{(s-\Delta s)}$ is zero $(\Rightarrow \Delta \dot{w} = 0)$ and $\ddot{w}_{(s-\Delta s)}$ is grater than zero (i. e.: $w_{(s)}$ has a minimum there), then $w_{(s)}$ is maximal at $s$. So we see that minima and maxima of velocity $w_{(s)}$ are following one another at distances $\Delta s$.

(3) If $|\Delta w| \, (\equiv |w_{(s)} - w_{(s-\Delta s)}|)$ is sufficiently small $|\Delta w| \ll 1$, one gets the linearized equation

$$\dot{w}_{(s)} + a \, \dot{w}_{(s-\Delta s)} = 0, \quad \text{(II, 3)}$$

which has been already discussed (as the non-relativistic limit) in Reference [1]. As was pointed out there, the general solution is of the form (final condition: $w_{(s \to \infty)} = 0$)

$$w_{(s)} = v_{(s)} \exp\left[\ln a \cdot \frac{s}{\Delta s}\right], \quad \text{(II, 4)}$$

where $v_{(s)}$ is arbitrary and restricted only by the periodicity requirement

$$v_{(s)} + v_{(s-\Delta s)} = 0. \quad \text{(II, 5)}$$

Obviously, we have to impose the condition

$$a = m_{\mathrm{el}}/m_{\mathrm{mech}} < 1, \quad \text{(II, 6)}$$

in order to get damped oscillations as solutions.

The preceding three points were rather self-evident. But now there arises the question, whether the amplitude of the approximate solutions (II, 4) becomes larger and larger in an unlimited manner, if one goes backward in time, or whether this amplitude is limited by the non-linear character of the exact equation (II, 1). Figure 1 shows a numerical solution of (II, 1), where one has started with the approximate solution (II, 4) for large $s$ $(v_{(s)} = 8 \sin(\pi[s/\Delta s - 8])$ assumed) and has then integrated backward in time. The flattening of the os-
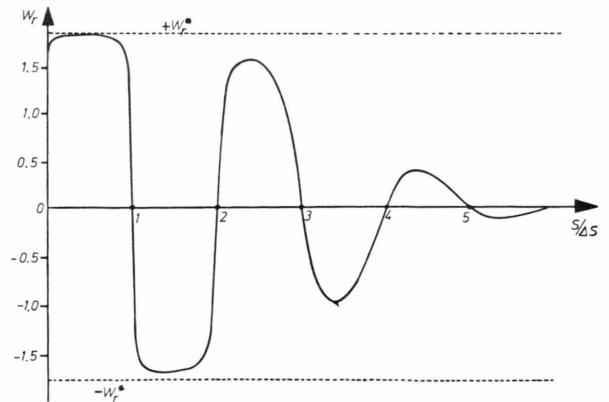


Fig. 1. Velocity $w_{(s)}$ versus proper time $s$ (in reduced units: $s/\Delta s$, where $\Delta s = 2/3 \, Z^2/m_{\mathrm{el}} \, c^2$). $w_{(s)}$ was prescribed in the interval $8 \leq s/\Delta s \leq 9$ as $v_{(s)} = 8 \sin(\pi[s/\Delta s - 8])$ [cf. Eq. (II, 4)]. In the distant past, $w_{(s)}$ approaches the constant value $-w_{\mathrm{r}}^*$ in the intervals $-(2n+1) < s/\Delta s < -2n$ $\{n = 0, 1, 2, 3, \ldots\}$ and $+w_{\mathrm{r}}^*$ in $-(2n+2) < s/\Delta s < -(2n+1)$. For the mass ratio we have chosen $m_{\mathrm{el}}/m_{\mathrm{mech}} = 0.3$.

cillations indicates the possibility of the boundedness of the amplitude, which shall be studied exactly in the next section.

## III. Exact Integration in the Force Free Case

The equation of motion (II, 1) can easily be integrated after some elementary manipulations.

First, one puts the equation of motion into the form

$$m_{\text{mech}} \left\{ \dot{w}_{(s)} - \dot{w}_{(s-\Delta s)} \right\} \tag{III, 1}$$
$$+ \dot{w}_{(s-\Delta s)} \left\{ m_{\text{mech}} + \tfrac{1}{2} m_{\text{el}} \left( e^{\Delta w} + e^{-\Delta w} \right) \right\} = 0 ,$$

or as well

$$\dot{w}_{(s-\Delta s)} \tag{III, 2}$$
$$+ \frac{2 \left( \dfrac{m_{\text{mech}}}{m_{\text{el}}} \right) \left( \dfrac{\mathrm{d}}{\mathrm{d}s} \Delta w \right) e^{\Delta w}}{\left( e^{\Delta w} + m_{\text{mech}}/m_{\text{el}} \right)^2 + \left[ 1 - (m_{\text{mech}}/m_{\text{el}})^2 \right]} = 0 .$$

Defining (observe $m_{\text{mech}} > m_{\text{el}}$)

$$\varrho = \frac{m_{\text{mech}}}{m_{\text{el}}} - \sqrt{\left( \frac{m_{\text{mech}}}{m_{\text{el}}} \right)^2 - 1} \tag{III, 3a}$$

$$1/\varrho = \frac{m_{\text{mech}}}{m_{\text{el}}} + \sqrt{\left( \frac{m_{\text{mech}}}{m_{\text{el}}} \right)^2 - 1} \tag{III, 3b}$$
$$(0 < \varrho < 1)$$

the left-hand side of Eq. (III, 2) can be written as a total derivative

$$\frac{\mathrm{d}}{\mathrm{d}s} \left\{ w_{(s-\Delta s)} + \frac{1/\varrho + \varrho}{1/\varrho - \varrho} \ln \left[ \frac{e^{\Delta w} + \varrho}{e^{\Delta w} + 1/\varrho} \right] \right\} = 0 . \tag{III, 4}$$

The integration constant is now chosen so that

$$w_{(s)} = w_\infty \quad \text{for} \quad \Delta w = 0 , \tag{III, 5}$$

i. e. *

$$w_{\mathrm{r}(s-\Delta s)} := w_{(s-\Delta s)} - w_\infty$$
$$= \sigma^{-1} \ln \left[ \left( 1 + \varrho\, e^{\Delta w} \right) / \left( e^{\Delta w} + \varrho \right) \right] \tag{III, 7}$$

where

$$\sigma = (1 - \varrho^2)/(1 + \varrho^2) . \tag{III, 8}$$

One concludes readily from (III, 7) that, if $w_{\mathrm{r}(s)}$ has a zero at $s = s^{(0)}$, then $\Delta w = w_{(s^{(0)} + \Delta s)} - w_{(s^{(0)})} = 0$ and this zero is reproduced at times $s^{(n)} = s^{(0)} + n \cdot \Delta s \; \{ n = 1, 2, 3, \ldots \}$:

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* Since the "solutions" of kind (III, 7) connect only the velocities $w_{(s)}$ and $w_{(s-\Delta s)}$ and these are of class $C^\circ [0, \infty)$ in any case, we are allowed in general to admit functions $w_{(s)}$ as solutions, which are only piecewise differentiable.

$$w_{(s^{(n)})} = w_\infty . \tag{III, 9}$$

Clearly, the quantity $w_\infty$ designates the "final" velocity, which is assumed by the particle in that case, where the oscillations are dying out completely with increasing time. The Lorentz invariance of the Eq. (III, 7) is now manifest, because the auxiliary variable $w$ changes by an additive constant under a unidirectional Lorentz boost.

We can study now our problem of Sect. II, whether there are unbounded deviations $w_{\mathrm{r}}$ from the equilibrium velocity $w_\infty$: Since the possible range of variation of $\Delta w$ is $-\infty < \Delta w < +\infty$, we conclude from (III, 7)

$$-w_{\mathrm{r}}^{(1)} < w_{\mathrm{r}(s-\Delta s)} < w_{\mathrm{r}}^{(1)} , \tag{III, 10a}$$

$$w_{\mathrm{r}}^{(1)} := \sigma^{-1} \ln (1/\varrho) . \tag{III, 10b}$$

So we see that the maximal amount of deviation of the velocity $w_{(s-\Delta s)}$ from its equilibrium value $w_\infty$ is finite. This means that, if $w_{(s)} \to \infty$, then $w_{(s-\Delta s)}$ remains finite. Hence, we conclude that if $w_{(s)} \to \infty$ is possible, this must happen at a finite time and not in the limit $s \to \infty$.

Let us first treat the case, where $w_{(s)}$ is bounded for all $s$. Then we can reformulate Eq. (III, 10) as

$$-w_{\mathrm{r}}^{(1)} < w_{\mathrm{r}(s)} < +w_{\mathrm{r}}^{(1)} \; \forall \, s . \tag{III, 11}$$

But this implies a further restriction on $\Delta w_{(s)}$:

$$\text{Max} \left\{ \Delta w^{(1)} \right\} = -\text{Min} \left\{ \Delta w^{(1)} \right\} \tag{III, 12}$$
$$= w_{\mathrm{r}}^{(1)} - \left( -w_{\mathrm{r}}^{(1)} \right) = 2\, w_{\mathrm{r}}^{(1)} .$$

These extremal values for $\Delta w_{(s)}^{(1)}$ are now substituted into (III, 7) in order to obtain a second bound for $w_{\mathrm{r}(s)}$:

$$-w_{\mathrm{r}}^{(2)} < w_{\mathrm{r}(s)} < +w_{\mathrm{r}}^{(2)} , \tag{III, 13}$$

where

$$w_{\mathrm{r}}^{(2)} = \sigma^{-1} \ln \left[ \frac{1 + \varrho \exp \left\{ -2\, w_{\mathrm{r}}^{(1)} \right\}}{\exp \left\{ -2\, w_{\mathrm{r}}^{(1)} \right\} + \varrho} \right] . \tag{III, 14}$$

Of course, we can iterate this restriction procedure, the first steps of which are illustrated in Fig. 2, where the inverse of (III, 7) has been plotted in the form

$$\Delta w = \sigma\, w_{\mathrm{r}}^{(1)} + \ln \left[ \frac{1 - \exp \left\{ \sigma (\hat{w}_{\mathrm{r}} - w_{\mathrm{r}}^{(1)}) \right\}}{\exp \left\{ (\hat{w}_{\mathrm{r}} + w_{\mathrm{r}}^{(1)}) \right\} - 1} \right] \tag{III, 15}$$
$$(\hat{w}_{\mathrm{r}} := w_{\mathrm{r}(s-\Delta s)}) .$$

The limit value $w_{\mathrm{r}}^*$

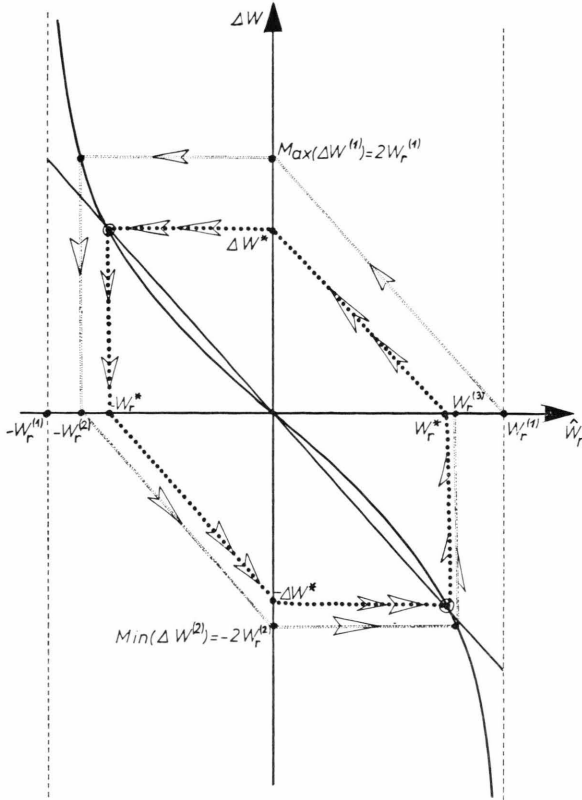$$w_{\mathrm{r}}^* = \lim_{n \to \infty} w_{\mathrm{r}}^{(n)} \quad (> 0) \tag{III, 16}$$

Fig. 2. The velocity difference $\Delta w$ is plotted versus the "advanced" velocity $\hat{w}_r$ as indicated by (III, 15) (solid, skew-symmetric curve). Observe that $\Delta w$ is only defined in the range $-w_r^{(1)} < \hat{w}_r < w_r^{(1)}$. The solid straight line through the origin follows from formula (III, 19). Its intersections with the curve (III, 15) determine the velocity bound $\pm w_r^*$. Starting the iterative procedure at $(w_r^{(1)}, 0)$ yields finally the closed polygon loop (doubly arrowed lines). The mass ratio was here chosen as $m_{\text{mech}}/m_{\text{el}} = 1.6$, which yields $w_r^* \approx 1.0097$.

of this iterative process is given by

$$w_r^* = \sigma^{-1} \ln \left[ \frac{1 + \varrho \exp\{-2\,w_r^*\}}{\exp\{-2\,w_r^*\} + \varrho} \right]. \quad \text{(III, 17)}$$

Because of

$$|\Delta w^*| \equiv 2\,w_r^*,$$
$$(\Delta w^* := \lim_{n \to \infty} \Delta w^{(n)} = 2\,w_r^*) \quad \text{(III, 18)}$$

we have to determine the intersection of the straight line

$$\Delta w = -2\,\hat{w}_r \quad \text{(III, 19)}$$

with the curve (III, 15) in Figure 2. The point of intersection determines graphically the accumulation point $w_r^*$ of the sequence $\{w_r^{(n)}\}$. Evidently,

there are two such points as solutions of (III, 17), which differ, however, only in sign according to maximal and minimal deviation $\pm \Delta w^*$. Figure 3 shows the velocity bound $w_r^*$ in terms of the mass ratio $m_{\text{mech}}/m_{\text{el}}$. In Fig. 1 we have already entered this velocity bound (dashed lines).
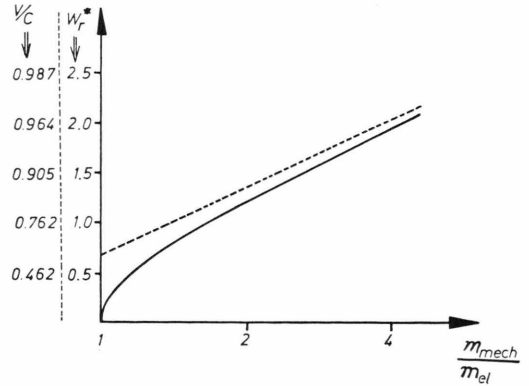


Fig. 3. Velocity bound $w_r^*$ versus mass ratio $m_{\text{mech}}/m_{\text{el}}$ (the latter one in a logarithmic scale). The dashed straight line follows from (III, 20). Observe that already for relatively small values of the mass ratio the velocity bound approaches the velocity of light $c\,(|\boldsymbol{v}^*|\,/\,c = \text{Tanh}\,w_r^*)$. For instance, one gets for $m_{\text{el}}/m_{\text{mech}} = 0.3 : w_r^* \approx 1.7875$ (see Fig. 1), which means that one can prescribe velocity changes up to $|\boldsymbol{v}|\,/c \approx 0.95$ in the first $\Delta s$-interval without generating the "acceleration catastrophe" (see Section IV).

An approximate solution of (III, 17) for $\varrho \to 0$ ($\Rightarrow \sigma \approx 1$) is readily obtained by putting

$$\exp\{w_r^*\} =: X,$$

which yields

$$\varrho\,X^3 - X^2 + X - \varrho = 0$$

with the asymptotic solution

$$X \approx \varrho^{-1} \Rightarrow w_r^* \approx \ln[2\,m_{\text{mech}}/m_{\text{el}}]. \quad \text{(III, 20)}$$

This approximation is also introduced in Fig. 3 (dashed straight line).

## IV. Stability of the Free Motion

We are now in a position to study the problem of stability of the free motion. Given the velocity $w_{(s)}$ in an initial interval of length $\Delta s$ ($= s_1 - s_0$, say)

$$w_{(s)} \equiv w_{\text{in}(s)}; \quad s_0 \leqq s \leqq s_1 \quad \text{(IV, 1)}$$

we can ask what is the further development of the motion?

First, we determine the equilibrium value $w_\infty$ of velocity by resolving Eq. (III. 7) with respect to this quantity

$$w_\infty = w_0 - \sigma^{-1} \ln \left[ \frac{1 + \varrho \exp\{\Delta w_{\text{in}}\}}{\varrho + \exp\{\Delta w_{\text{in}}\}} \right], \quad (\text{IV}, 2)$$

where we have put

$$w_0 := w_{\text{in}(s=s_0)} \quad (\text{IV}, 3)$$

$$\Delta w_{\text{in}} := w_{\text{in}(s=s_1)} - w_{\text{in}(s=s_0)} =: w_1 - w_0.$$

For all continuous functions $w_{\text{in}(s)}$ the equilibrium velocity $w_\infty$ is well-defined, because it depends only from the boundary values $w_0$ and $w_1$. One has

$$w_\infty \rightarrow w_0 - \sigma^{-1} \ln \varrho = w_0 + w_r^{(1)} \quad \text{for} \quad \Delta w_{\text{in}} \rightarrow +\infty,$$

$$w_\infty \rightarrow w_0 - w_r^{(1)} \quad \text{for} \quad \Delta w_{\text{in}} \rightarrow -\infty. \quad (\text{IV}, 4)$$

Since $w_\infty$ as a function of $\Delta w_{\text{in}}$ ($= w_1 - w_0$) is monotonically increasing (with $w_\infty = w_0$ for $\Delta w_{\text{in}} = 0$), the equilibrium velocity $w_\infty$ is situated between $w_0$ and $w_1$

$$w_0 \leqq w_\infty \leqq w_1 \quad \text{for} \quad \Delta w_{\text{in}} > 0,$$

$$w_0 \geqq w_\infty \geqq w_1 \quad \text{for} \quad \Delta w_{\text{in}} < 0,$$

$$w_\infty = w_0 \pm w_r^* \quad \text{for} \quad \Delta w_{\text{in}} = \pm 2 w_r^*,$$

$$w_\infty = w_0 \quad \text{for} \quad \Delta w_{\text{in}} = 0. \quad (\text{IV}, 5)$$

Obviously, we have now to discern between three cases, which lead to a qualitatively quite distinct behaviour of the free particle:

## (α)

The initial velocity $w_{\text{in}(s)}$ satisfies the condition

$$w_\infty - w_r^* < w_{\text{in}(s)} < w_\infty + w_r^* \quad (\text{IV}, 6)$$

for all $s$ contained in the initial interval $s_0 \leqq s \leqq s_1$. Then it is easy to show that

$$\lim_{s \to \infty} w_{(s)} = w_\infty, \quad (\text{IV}, 7)$$

because we can choose an arbitrary $s_A$ out of the initial interval, designate the according value of $w_{\text{in}(s)}$ with $w_{r,A} = w_{\text{in}(s_A)} - w_\infty$, and see from Fig. 4 that $\Delta w_A = w_{(s_A + \Delta s)} - w_{(s_A)}$ satisfies the inequality (observe sign $\Delta w_A \neq$ sign$[w_{(s_A)} - w_\infty]$)

$$|\Delta w_A| < 2 |w_{r,A}|, \quad (\text{IV}, 8)$$

so that

$$|w_{(s_A + \Delta s)} - w_\infty| < |w_{(s_A)} - w_\infty|. \quad (\text{IV}, 9)$$

We can continue this procedure $[w_{r,A} (\equiv 1) \rightarrow w_{r(s_A + \Delta s)} (\equiv 2) \rightarrow 3 \rightarrow 4 \ldots \rightarrow$ origin of the dia-



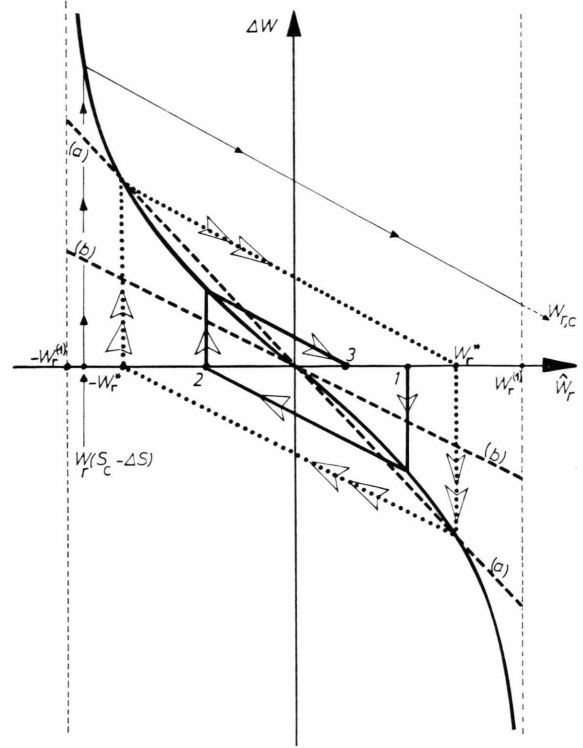Fig. 4. The curve (III, 15) is plotted again with the closed polygon loop (double arrow lines, dotted). According to case ($\alpha$), one starts at point 1. Since the curve (III, 15) is always situated between $\Delta w = -2 \hat{w}_r$ [straight line (a)] and $\Delta w = -\hat{w}_r$ [straight line (b)], the inequalities (IV; 8, 9) follow from this construction. Observe, that all inclined lines of the iterative construction are parallel to line (b).

gram] and find indeed (IV, 7). This argument is applicable to all possible choices $s_A$ out of the initial interval. The solution exhibited in Fig. 1 is of this kind.

One can easily find the limits of the rapidity, with which the oscillations are damped down: Resolving (III, 7) with respect to $w_{r(s)}$ yields

$$\exp(w_r) = \frac{1 - \varrho \exp\{\sigma \hat{w}_r\}}{\exp\{\sigma \hat{w}_r\} - \varrho} \cdot \exp\{\hat{w}_r\}, \quad (\text{III}, 7')$$

which can also be written as

$$\exp\{w_r + (\gamma - 1)\hat{w}_r\} = \frac{1 - \varrho \exp\{\sigma \hat{w}_r\}}{\exp\{\sigma \hat{w}_r\} - \varrho}$$
$$\cdot \exp\{\gamma \hat{w}_r\}, \quad (\text{III}, 7'')$$

where $\gamma$ has chosen to be

$$\gamma = \frac{(1 + \varrho)^2}{1 + \varrho^2} = 1 + \frac{m_{\text{el}}}{m_{\text{mech}}}.$$

Now put here $\exp\{\hat{w}_r\} =: y$ and consider the function $f_{(y)}$

$$f_{(y)} = \frac{1 - \varrho\, y^\sigma}{y^\sigma - \varrho} \cdot y^\gamma$$

with derivative

$$f'_{(y)} = -\varrho\,\gamma\,\frac{y^{\gamma-1}(1-y^\sigma)^2}{(y^\sigma - \varrho)^2}.$$

Because of $y > 0$ we have $f'_{(y)} < 0$, and this implies (observe $f_{(1)} = 1$)

$$y \leqq 1 \Rightarrow f_{(y)} \geqq 1\,,$$
$$y \geqq 1 \Rightarrow f_{(y)} \leqq 1\,.$$

Hence, we conlude for (III, 7″)

$$\hat{w}_r > 0 \Rightarrow w_r < -(m_{el}/m_{mech}) \cdot \hat{w}_r\,,$$
$$\hat{w}_r < 0 \Rightarrow w_r > -(m_{el}/m_{mech}) \cdot \hat{w}_r\,,$$

which means for the absolute values

$$|w_r| \geqq (m_{el}/m_{mech})\,|\hat{w}_r|\,,$$

where this is an equality for the non-relativistic limit (II, 3). The last equation determines the maximal damping rapidity.

### (β)

The initial velocity $w_{in(s)}$ satisfies the condition

$$w_\infty - w_r^* \leqq w_{in(s)} \leqq w_\infty + w_r^*\,, \quad (IV, 10)$$

which is understood to mean that there is at least one value $\tilde{w}_{in}$ of $w_{in(s)}$ such that

$$|\tilde{w}_{in} - w_\infty| = w_r^*\,. \quad (IV, 11)$$

All other values of $w_{in(s)}$, for which (IV, 11) is not satisfied but rather (IV, 6), can be treated analogous to case (α) with the same results found there. But the values of character $\tilde{w}_{in}$ are mapped alternatively into the points $\pm w_r^*$ on the $\hat{w}_r$-axis of Fig. 4 (double arrow lines). Since the neighbouring values of $\tilde{w}_{in}$ tend to $w_\infty$ for $s \to \infty$, there must evolve spikes of constant amplitude $(w_r^*)$, which
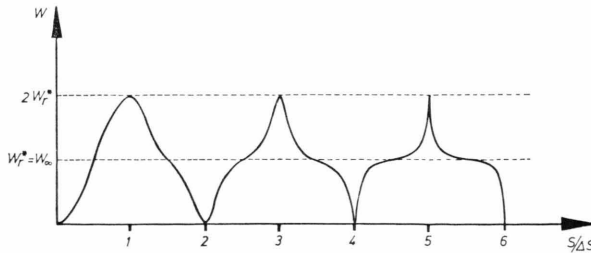


Fig. 5. Undamped velocity oscillations for $m_{el}/m_{mech} = 0.5$ $\Rightarrow w_\infty = w_r^* \approx 1.2244$. The initial velocity in $0 \leqq s/\Delta s \leqq 1$ was prescribed to be $w_{in(s)} = w_r^*\,\{1 - \cos[\pi\,(s/\Delta s)]\}$.

become steeper and steeper with increasing time. Figure 5 exhibits a plot of this sort of solution.

### (γ)

The initial velocity $w_{in(s)}$ is such that there occur values $\overline{w}_{in}$, which satisfy

$$w_r^{(1)} > |\overline{w}_{in} - w_\infty| > w_r^*\,. \quad (IV, 12)$$

In this case, one can show that the velocity $w_{(s)}$ becomes infinite (the particle assumes the velocity of light $c$) at a certain finite proper time ($s_\infty$, say). At this time $s_\infty$ the world-line of the particle ends.

We can choose again a time ($s_B$, say), where $w_r^{(1)} > |w_{in(s_B)} - w_\infty| > w_r^*$. Then we can determine $\Delta w_B = w_{(s_B + \Delta s)} - w_{(s_B)}$ from Fig. 4, where now

$$|\Delta w_B| > 2\,|w_{r(s_B)}| \quad (IV, 13)$$

with $\text{sign}\,[\Delta w_B] \neq \text{sign}\,[w_{r(s_B)}]$. Hence

$$|w_{(s_B + \Delta s)} - w_\infty| > |w_{(s_B)} - w_\infty| \quad (IV, 14)$$

and the velocity after proper time intervals of length $\Delta s$ is ever increasing in amount. Finally, there is a certain time ($s_c$, say), where we may have $w_{r(s_c)} =: w_{r,c} > w_r^{(1)}$ well-definded but $w_{r(s_c + \Delta s)}$ does no longer exist (see Fig. 4), because $s_c + \Delta s > s_\infty$. The values of $w_{(s_c - n\Delta s)}$ $\{n = 1, 2, 3, \ldots\}$ can be followed back in time by the construction of Figure 4. They tend to $w_\infty \pm w_r^*$, and if we now think again forward in time, we see that an infinitely small deviation of velocity $w_{in}$ from $w_\infty \pm w_r^*$ (such that $|w_{in} - w_\infty| > w_r^*$) leads to the acceleration catastrophe just described [*].

So we see that we can have "runaway solutions" in some sense also in this new non-local theory, contrary to the first purely electromagnetic non-local theory [4], where $m_{mech}$ was set equal to zero. However, one realizes on account of (II, 6) that the former, purely electromagnetic model [4] is unstable with respect to the addition of a mechanical mass term with small mass $m_{mech}$. But in comparison to the runaway solutions of the Lorentz-Dirac theory one has gained a certain *stability of the free motion* of the radiating electron: Only if the particle is pushed to velocity changes, which equal (roughly) the velocity of light within a $\Delta s$-interval, the above

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[*] We want to stress, however, that the proper time value $s_\infty$ must not necessarily correspond to a finite lab time $t_\infty$; rather one wold except that the particle will assume the velocity of light ($w = \infty$) not until the infinitely distant (lab) future, according to $c\,t_\infty = \int_0^{s_\infty} \text{Cosh}\,w_{(s')}\,ds'$.

mentioned instability arises. In all other cases, where physically reasonable forces are applied, the particle oscillates down to constant velocity after the end of the force (see for instance the constant-force problem in the next section).

## V. The Constant-force Problem and Causality Violation

Now we proceed to study the motion of the particle in a constant, homogenious external force field of finite range, i. e. we have the equation of motion

$$\dot{w}_{(s)} + a\,\dot{w}_{(s-\Delta s)}\,\mathrm{Cosh}\,\{w_{(s)} - w_{(s-\Delta s)}\} = Q_{(s)} \quad (V,1)$$
$$(Q_{(s)} := K_{(s)}/m_{\mathrm{mech}}\,c^2)$$

where

$$Q_{(s)} = \begin{cases} 0; & s < 0; \\ Q_0 = \mathrm{const}; & 0 \leqq s < l; \\ 0; & s \geqq l. \end{cases} \quad (V,2)$$

The analogous investigations for the Lorentz-Dirac theory have been performed by Rohrlich[5] and for the purely electromagnetic model in a previous paper[6]. So we shall concentrate on a comparison of the results of the present model with those of the



Fig. 6. Velocity $w_{(s)}$ for the constant force $Q_0\,\Delta s = 2.2$; $l = 6\,\Delta s$; and $m_{\mathrm{el}}/m_{\mathrm{mech}} = 0.6$. Since the acceleration in the first interval $(0 \leqq s/\Delta s \leqq 1)$ is determined only by $m_{\mathrm{mech}}$, it is greater than the corresponding value (dashed line) for a non-radiating particle of same total mass $m_{\mathrm{exp}} = m_{\mathrm{mech}} + m_{\mathrm{el}}$. After the end of the force the particle performs damped oscillations around the equilibrium value $w_\infty$.

Lorentz-Dirac theory as far as their solutions for the special force field $(V, 2)$ are concerned.

Figure 6 exhibits the "velocity" $w_{(s)}$ for the present equation of motion $(V, 1)$. Let us first turn to the phenomenon of causality violation. We know that as well the Lorentz-Dirac theory[5] as the purely electromagnetic model[6] exhibit causality violation. But the present model does obviously not have to struggle against these unphysical effects. We can prescribe the velocity $w_{(s)}$ in the initial interval $-\Delta s \leqq s \leqq 0$ arbitrarily and then compute from the equation of motion $(V, 1)$ the motion in the neighbouring interval of length $\Delta s$. For Fig. 6 we have chosen $w \equiv 0$ in the initial interval $-\Delta s \leqq s \leqq 0$. Since the derivative $\dot{w}_{(s)}$ for $s \geqq 0$ is only dependent from the force $K_{(s)}$ and the velocity $w_{(s)}$ at the *same* time and from the *past* values of velocity and acceleration, it is clear that neither the future forces nor the future kinematics of the world line have an influence on the present state of the motion. Hence, there is no causality violation in the present model including a mechanical mass term.

These statements must be considered as a hard attack against the Lorentz-Dirac equation

$$m\,c^2\,\dot{u}^\lambda = K^\lambda + \tfrac{2}{3}\,Z^2\,[\ddot{u}^\lambda + (\dot{u}\,\dot{u})u^\lambda]\,, \quad (V,3)$$
$$(m = m_{\mathrm{mech}} + m_{\mathrm{el}})$$

because this equation is obtainable by a simple power-series expansion of the non-local quantities in the exact equation (see Reference[2]). These expansions are just of the same kind as used by Dirac[7], Rohrlich[5], or Teitelboim[8] in order to "deduce" the Lorentz-Dirac equation from Maxwell's theory. However, these authors have worked a priori with power-series expansions, and therefore the impression could arise[9] that causality violation as a non-local effect be an intrinsic property of electromagnetic interactions. But this reasoning can hardly by accepted, because the present example shows very clearly that causality violation can also be a consequence of a faulty perturbation expansion: the original unexpanded equation $(I, 1)$ does not exhibit this unphysical effect!
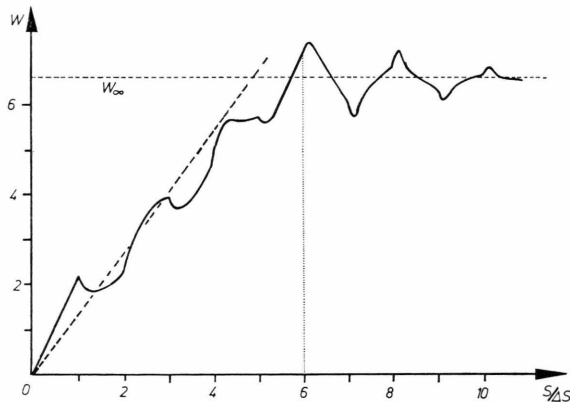
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